PEAKING OF THE PRESSURE PULSE IN FLUID-FILLED TUBES OF SPATIALLY VARYING COMPLIANCE

A. C. L. BARNARD, W. A. HUNT, W. P. TIMLAKE, *and* E. VARLEY
From the IBM Scientific Center, Houston, Texas

ABSTRACT Calculations are made for a fluid-filled tube with characteristics approximately those found physiologically. The pressure variation, diameter, and compliance at the input end are as measured by Lawton for the abdominal aorta of a dog. After a 30 cm-long input section of constant k ($=dp/dA$), the tube is taken to stiffen by approximately the amount measured by Patel et al., i.e., k increases by a factor of 5 over the next 40 cm. The cross-section remains constant. Pressure and velocity wave forms are calculated at 8 stations spaced at 10-cm intervals down the tube. The pressure pulse leading edge is found to become steeper in the stiffening section. The peak height of the pressure pulse is found to increase by about 50% and the velocity pulse to decrease by about 30% as the disturbance propagates over a distance of 70 cm. These values agree qualitatively with the experimental physiological values given by McDonald. Most of the pressure peaking takes place upstream of the stiffening section.

INTRODUCTION

According to McDonald, "the remarkable change in the shape of the pressure wave in the arteries as it travels from the heart to the periphery has provided a most intriguing problem in circulatory physiology since the first introduction of adequate manometers over fifty years ago" (1). The peak value of the pressure pulse increases and the leading edge of the pulse becomes steeper as the pulse propagates from the proximal to the peripheral part of the arterial system. In contrast, the peak value of the flow pulse decreases markedly and its leading edge becomes less steep. This behavior differs from that observed in a simple rubber tube filled with viscous fluid, in which case both pressure and velocity pulses are attenuated. The latter is the effect predicted by Womersley's theory (2) of pulse transmission, in an infinitely long tube, which therefore fails to account for the behavior observed physiologically.

Several reasons have been proposed for the physiological behavior. McDonald, following Womersley, attributed the effect to reflections from discrete sites such as the bifurcation of the aorta. The reflections of one pulse were taken to be superimposed on later pulses.[1] However, there is some experimental evidence that discrete reflections are unimportant in the arterial system (3, 4).

Attempts at mathematical analysis of blood flow have been reviewed by Fox and Saibel (5) who concluded that the neglect of the nonlinear terms in the equations, and of the tube taper, were serious defects in the work published up to that time.

The propagation of a pulse into a quiescent region of fluid in a tube can be treated by the nonlinear theory of wave-front propagation developed by Varley and Cumberbatch (6). In the first section of the present paper, an ordinary differential equation will be derived which relates the velocities and pressures at the front during propagation. This will suggest a simple physical system in which the leading edges of the pressure and velocity pulses behave as they do physiologically. The second section of the paper will be devoted to numerical calculations which confirm the predicted behavior of the leading edges and also show that the peak values behave as they do physiologically.

NONLINEAR THEORY OF WAVE-FRONT PROPAGATION

In a previous paper (7) equations were obtained to describe a system consisting of a viscous fluid in a compliant tube. The equation of motion for the fluid was given as:

$$\frac{\partial U}{\partial t} + \alpha U \frac{\partial U}{\partial z} + (1 - \alpha) \frac{U}{A} \frac{\partial A}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{\gamma U}{A} \tag{1}$$

to which must be added the continuity equation

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z} (UA) = 0. \tag{2}$$

For the case of interest, pressure variations and area variations are proportional to one another with little phase difference (8), so

$$p = k \, \Delta A \tag{3}$$

where

$$\Delta A = A - A_0 \quad \text{and} \quad k = \tfrac{2}{3} E \left(\frac{h_0}{R_0}\right) \frac{1}{A_0},$$

where E is Young's Modulus and h/R is the ratio of wall thickness to tube radius. It is not difficult to carry through the following treatment with A_0 varying with distance (static tube taper) and with k varying with distance but A_0 constant (tube with changing compliance).

There is wide agreement that the arteries at the periphery are stiffer than the proximal arteries (8). The aorta, for instance, is usually regarded as an elastic

[1] Pages 203 and 232 of reference 1.

reservoir. For this reason the following development will be made for a tube with no static taper (although there will be dynamic taper caused by the pressure variation) but which changes in compliance along its length. At the end the results for a tapering tube will be stated without proof. Equations 1 and 2 were written in terms of the set of independent variables (z, t). If the equation of the front is $\beta(z, t) = z - ct = 0$ where c is not necessarily constant, constant β implies a point moving so as to keep a fixed relationship to the front. A new set of independent variables (z, β) may be used.

This simply corresponds to specifying a point not by its distance and time, but by its distance and relationship to the front. If $(dz, d\beta)$ and (dz, dt) are corresponding changes in the two sets of variables, and produce a change in some dependent variable such as U, then

$$dU = \left(\frac{\partial U}{\partial z}\right)_t dz + \left(\frac{\partial U}{\partial t}\right)_z dt = \left(\frac{\partial U}{\partial z}\right)_\beta dz + \left(\frac{\partial U}{\partial \beta}\right)_z d\beta$$

if β is held constant (traveling along with the front) then

$$\frac{D}{Dz} = \left(\frac{\partial}{\partial z}\right)_\beta = \left(\frac{\partial}{\partial z}\right)_t + \frac{dt}{dz}\left(\frac{\partial}{\partial t}\right)_z \tag{4}$$

where the first quality defines the notation.

Ahead of the front $(\beta > 0)$ the fluid is assumed to be stagnant in a nontapering tube, so that

$$U = p = 0 \qquad A = \text{Constant}$$

and

$$\frac{DU}{Dz} = \frac{Dp}{Dz} = \frac{DA}{Dz} = 0.$$

These relationships persist as $\beta \to 0$ from the positive side. At $\beta = 0$, using equation 4,

$$\left(\frac{\partial}{\partial z}\right)_t = -\frac{1}{c}\left(\frac{\partial}{\partial t}\right)_z \tag{5}$$

where this operator may be applied to U, A, or p. Here c is written for dz/dt.

For $\beta > 0$, equations 1 and 2 become (writing A_t for $\partial A/\partial t$, etc.)

$$A_t + AU_z = 0 \tag{6}$$

$$U_t + p_z = 0. \tag{7}$$

Equations 6 and 7 continue to hold as $\beta \to 0$ and using equations 4 and 5 they become

$$A_t - \frac{A}{c} U_t = 0 \tag{8}$$

$$-\frac{k}{c}A_t + U_t = 0. \tag{9}$$

Equations 8 and 9 are two homogeneous equations in two unknowns. From equation 8, $A_t/U_t = A/c$ and from equation 9, $A_t/U_t = c/k$. The equations are consistent only if

$$c^2 = kA. \tag{10}$$

An equivalent statement is that the homogeneous equations 8 and 9 have a nontrivial solution only if the determinant of the coefficients is zero, that is if

$$\begin{vmatrix} 1 & -A/c \\ -k/c & 1 \end{vmatrix} = 0$$

which again gives equation 10. This defines the propagation speed c, which will be a function of z if k is a function of z. Equations 8 and 9 may now be rewritten as

$$p_t/U_t = (kA)^{1/2} = c \tag{11}$$

Equation 11 relates the rates of change of pressure and velocity at the front. As expected, the derivatives are of the same sign, since a positive-going pressure pulse will be associated with a positive-going velocity pulse. Down the tube, A may be taken as almost constant for fairly stiff tubes without static taper. Changes in k give rise to the possibility that the leading edge of the pressure pulse can get steeper, relative to the velocity pulse. This can be emphasized by differentiating equation 11:

$$\frac{D}{Dz}p_t = c\frac{D}{Dz}U_t + U_t\frac{D}{Dz}c \tag{12}$$

From equation 12 it can be seen that moving along with the front the velocity pulse leading edge may be getting less steep ($(D/Dz)U_t$ negative) while the pressure pulse leading edge may be getting steeper ($(D/Dz)p_t$ positive). This condition will exist if Dc/Dz is positive and sufficiently large (since $U_t > 0$ for the cases of interest). Thus, in a stiffening tube it appears possible to generate behavior like that observed physiologically.

However equation 12 expresses only the relative behavior of p_t and U_t and it is necessary to obtain the behavior of one of them separately. The partial derivatives of equations 1 and 2 with respect to time are formed. Using equations 3, 4, and 11 and placing $U = 0$ ahead of the front, the following equations hold for $\beta \geq 0$:

$$U_{tt} - \frac{k}{c}A_{tt} = -\frac{\gamma}{A}U_t + \frac{(2\alpha - 1)}{c}U_t^2 - \frac{A}{c}U_tk_z - k\frac{D}{Dz}\left(\frac{A}{c}U_t\right) \tag{13}$$

$$-\frac{A}{c}U_{tt} + A_{tt} = \frac{2A}{c}U_t^2 - A\frac{D}{Dz}U_t. \tag{14}$$

The coefficient matrix for the left-hand sides of equations 13 and 14 is the same as that obtained previously, and its determinant is therefore zero. From the right-hand sides, multiplied by the appropriate factors, the following differential equation is obtained:

$$2c \frac{D}{Dz} U_t = \left(-\frac{Dc}{Dz} - \frac{\gamma}{A}\right) U_t + \frac{(2\alpha + 1)}{c} U_t^2 \tag{15}$$

and using equations 12 and 15

$$2 \frac{D}{Dz} p_t = \left(\frac{Dc}{Dz} - \frac{\gamma}{A}\right) U_t + \frac{(2\alpha + 1)}{c} U_t^2. \tag{16}$$

From equations 15 and 16 it is clear that for sufficiently large and positive Dc/Dz the quantity p_t at the front will increase, while U_t decreases, since the derivative appears with opposite sign in the two equations.

Approximate numbers for the physiological situation can be inserted on the right-hand sides of equations 15 and 16. For the larger arteries in a dog (8), A is of the order of 1 cm^2; k is of the order of 10^6 dyne/cm^4 and increases by a factor of about 12 between the ascending and abdominal aorta (8). Thus, c is apporximately 10^3 cm/sec and Dc/Dz about 20 cm/sec per centimeter. The velocity pulse rises to about 100 cm/sec in 1/20 sec (1) so that $U_t \approx 2000$ cm/sec^2. If a parabolic velocity profile is assumed, $\gamma \approx 1$ cm^2/sec for blood. It is clear that the term Dc/Dz is substantially larger than the other terms on the right-hand sides of equations 15 and 16. The conclusion is unchanged by wide deviations from the above estimates of the quantities. Thus, in a typical physiological situation the theory predicts that the leading edge of the velocity pulse will become less steep and the leading edge of the pressure pulse more steep.

It must be noted that the above analysis reveals only the behavior of P_t and U_t just at the front. It has *not* been shown that the pressure pulse peak height increases as the disturbance propagates, since this involves conditions behind the front which are not treated above.

The derivation for a tube which both tapers and stiffens is very smiliar to that given above. Equation 12 is unchanged and the equation which corresponds to equation 16 is

$$2 \frac{D}{Dz} p_t = \left(\frac{c}{2} \frac{D}{Dz} \ln \frac{k}{A} - \frac{\gamma}{A}\right) U_t + \frac{(2\alpha + 1)}{c} U_t^2. \tag{17}$$

It is straightforward, though tedious, to carry through the derivation for the more general case where the fluid ahead of the front is in a steady-flow condition, rather than a zero-flow condition as above. The results are given in the appendix.

NUMERICAL CALCULATIONS

The previous section was an analytical prediction of the behavior of solutions to partial differential equations 1 and 2 in certain circumstances, namely a wave front

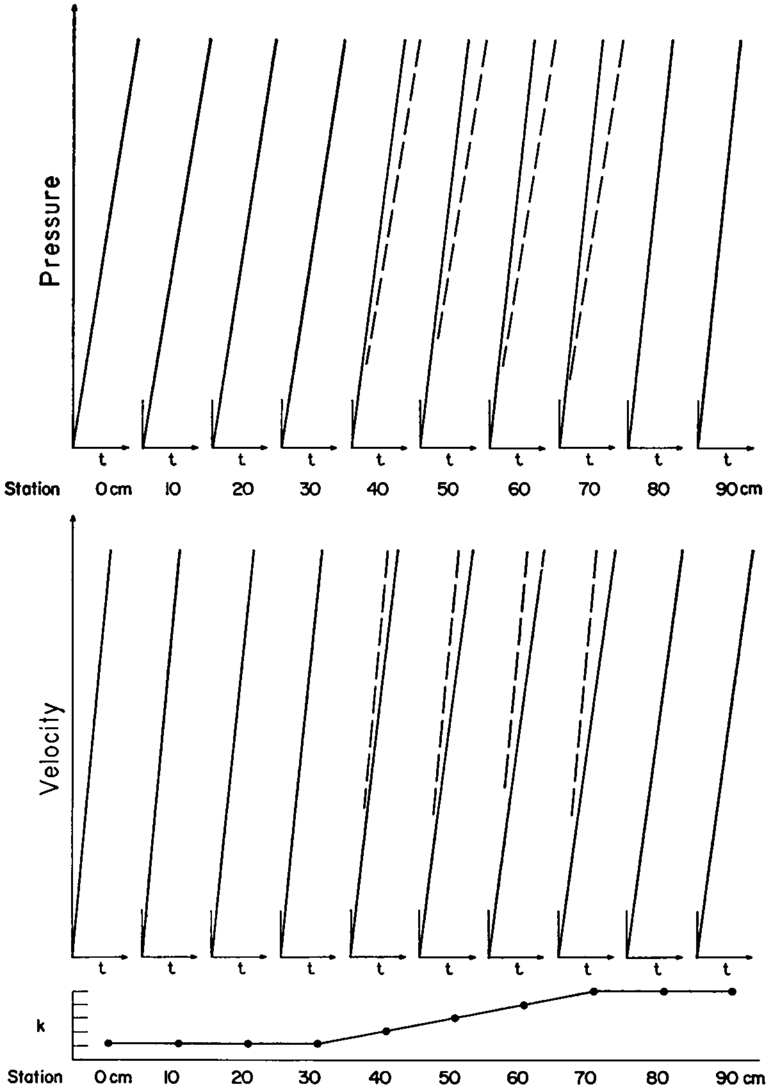

FIGURE 1 (lower) The tube stiffness, k, as a function of distance down the tube. The dots represent the values at stations spaced by 10 cm down the tube. Note that the stiffness increases by a factor of 5 between the 30- and 70-cm stations. The tube cross-section remains constant.

(upper) The solid lines are the leading edges of the pressure and velocity pulses plotted against time at each of the stations. The dashed lines are repeats of the leading edge at the 0-cm station. By comparing the dashed and solid lines, it can be seen that in the stiffening section of the tube, the pressure pulse leading edge becomes more steep and the velocity pulse leading edge less steep.

propagating into quiescent conditions. The predictions may be tested by numerical calculations with equations 1 and 2 using the methods outlined in a previous paper (7). Such calculations give the behavior at all points, not only at the front.

The input data used consisted of the pressure pulse measured by Lawton in the abdominal aorta of a dog. Womersley[2] quoted Lawton's experimental results and made a Fourier analysis of the pulse. Lawton measured the dynamic diameter of the vessel, so that the appropriate values of the quantities k and A_0 were known and were used for the input. The tube was taken to have constant k for the first 30 cm, then k increasing linearly by a factor 5 over the next 40 cm and thereafter remaining constant. (Patel, Greenfield, and Fry (8) found a factor of approximately 12 increase in k between the ascending aorta and the abdominal aorta in a dog.) The tube was taken to be very long (12 m) to avoid reflections from the end. In fact, the calculation was terminated when the pulse arrived at the distal end.

Fig. 1 shows part of the results of one calculation in which time zero was taken at a steeply rising part of the pressure pulse to give a sharp leading edge. The leading edges of the pressure and velocity pulses are shown at successive stations down the tube. It can be seen that the wave fronts remain almost constant in steepness through the 30 cm–long input section of the tube. In the stiffening section the pressure pulse leading edge becomes more steep and the velocity pulse leading edge becomes less steep. The predictions of the Varley-Cumberbatch theory are therefore confirmed numerically.

Fig. 2 and Table I show the results of another calculation in which time zero was chosen at about the minimum of the pressure curve. Since the pressure throughout the tube is set to the proximal pressure at time zero, the effect of this is to give a larger time-averaged driving pressure than in the previous calculation. It can be seen that the "peaking" effects observed physiologically have been reproduced. The pressure pulse at successive stations down the tube becomes higher and narrower, whereas the velocity pulse becomes lower and wider. This is emphasized in Fig. 3, where the pulse at the 70 cm station is superimposed on the input (0 cm) pulse. The calculations were continued through two cycles of the input pulse. The first pulse propagates into stagnant conditions, since the velocities and pressures are set initially to zero. The second pulse propagates into slightly different conditions. The use of a nonstiffening input section brings out an interesting effect: the pulse begins to peak upstream of the point where stiffening begins. It is possible to regard this as an effect of partial reflection of the disturbance from the stiffening region downstream. If this view is adopted, the effect consists of modification of a pulse by partial reflection of *itself*.

In the calculations reported in this paper, reasonable values of the parameters α and γ were used, based on the experience gained in making the catheter calcula-

[2] Page 52 of reference 2.

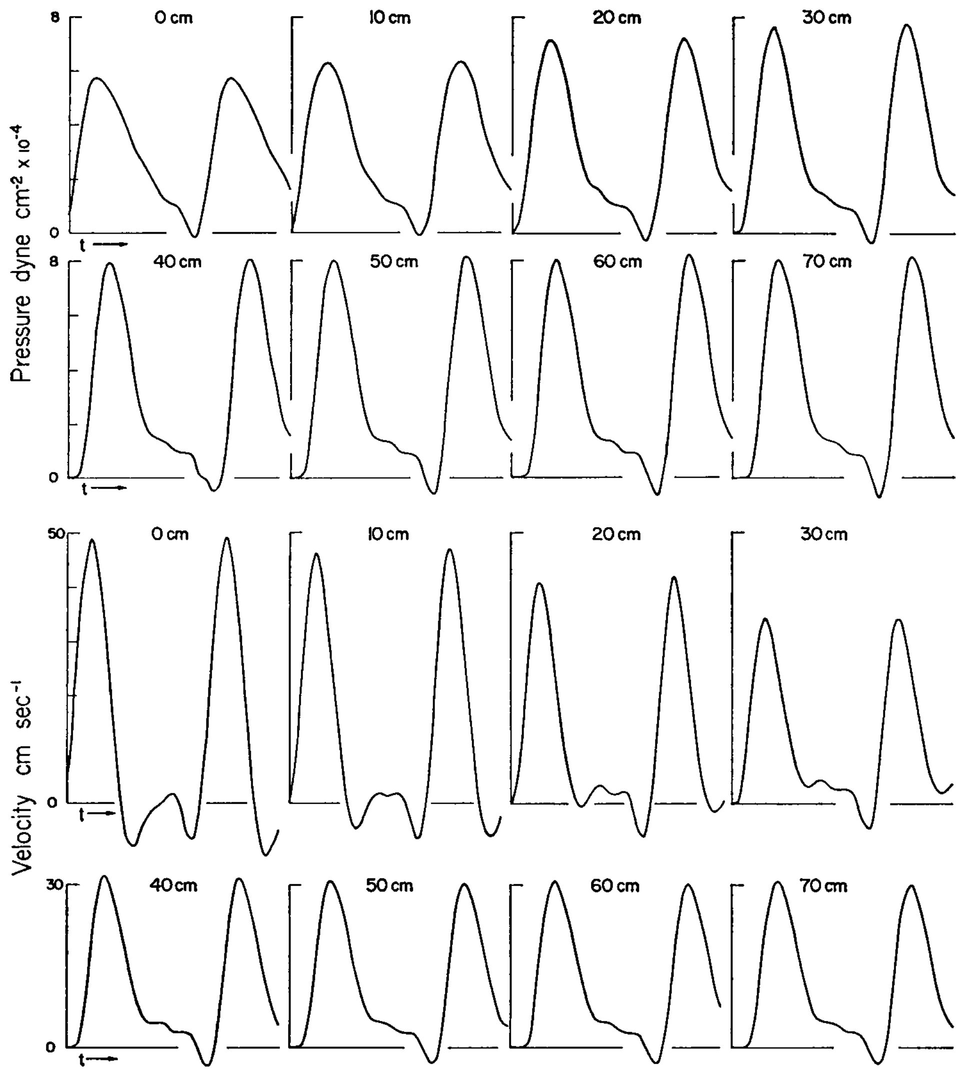

FIGURE 2 Pressures and velocities as a function of time at eight stations down a 12 m–long tube. The 0-cm station is the input, where the pressure is specified. All other pressures and all velocities are calculated. The pressure is measured above a base pressure. The time period of the input pulse is 0.275 sec. The tube stiffens by a factor 5 linearly between the 30- and 70-cm stations, but remains constant in cross-section (see Fig. 1, lower).

tions reported in the previous paper (9), i.e., $\alpha = 1.33$ $\gamma = 0.8$ cm^2/sec. Both parameters were found to have only a minor effect on the results. Increasing γ to 1.6 produced about 1% less pressure peaking. Decreasing α to zero reduced the peaking by about 1% also.

TABLE I

NUMERICAL VALUES OF PRESSURES AND VELOCITIES

Distance down tube	Peak pressure (dyne/cm^{-2})		Peak velocity (cm/sec^{-1})	
	1st cycle	2nd cycle	1st cycle	2nd cycle
cm				
0	57819	57821	47.99	48.90
10	63399	63771	45.93	46.67
20	71271	72061	40.80	41.22
30	76744	77849	34.03	34.02
40	79214	80444	31.34	31.14
50	80194	81507	30.61	30.34
60	80428	81759	30.50	30.21
70	80358	81691	30.52	30.24
80	80268	81598	30.48	30.20
90	80184	81506	30.45	30.17

γ = 0.8 cm^2/sec α = 1.33 (dimensionless).
k = 10^6 (proximal) dyne/cm^4. Time period = 0.275 sec.
Calculation dated 17 March, 1966.

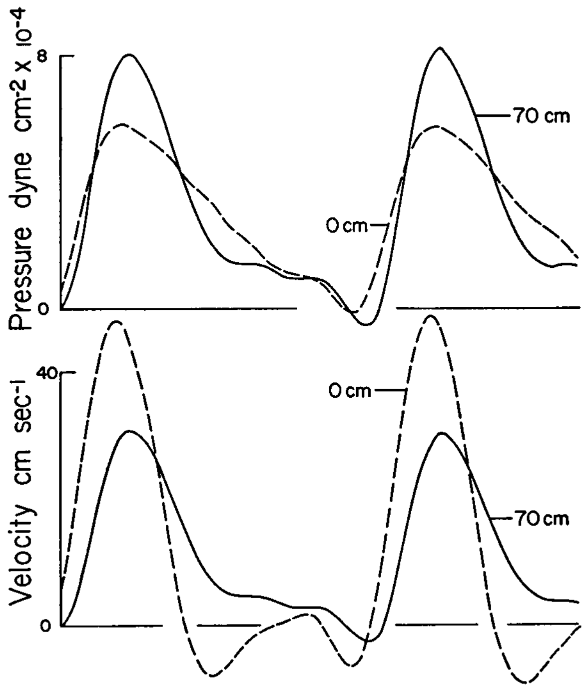

FIGURE 3 Pressure and velocity pulses at the 0- and 70-cm stations taken from Fig. 2 and superimposed for ease of comparison. The delay in the 70-cm curves, due to the finite propagation speed of the disturbance, has been removed in plotting this curve.

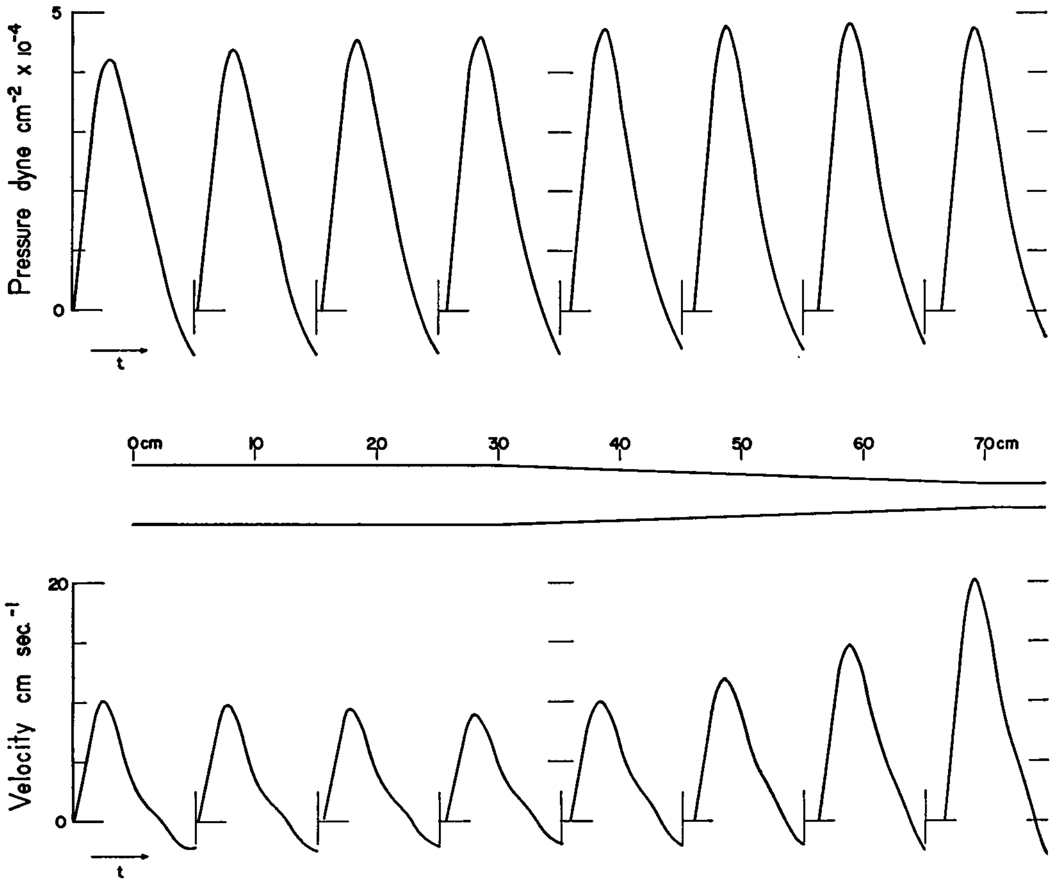

FIGURE 4 Pressures and velocities for a tube which constricts in cross-section by 60% between the 30- and 70-cm stations, with k remaining constant. The geometric taper is indicated graphically in the center of the figure.

Some calculations have been made with tapering tubes. Although the pressure pulse peaks up in a constricting tube, it is obvious that to satisfy the continuity equation the velocity pulse will do likewise. This is illustrated in Fig. 4, which shows the results of a calculation with the tube constricting to 40% of its original cross-section linearly over a length of 40 cm, with k remaining constant.

APPENDIX

PROPAGATION OF DISTURBANCE INTO STEADY FLOW CONDITIONS

Rather than assume that the pulse is propagated into a stagnant region, i.e. $U = 0$; $A =$ constant, we assume here that there is steady flow, i.e. $U(z)\ A(z) = \kappa$ (a constant), ahead of the wave front. Under the more general assumption, the necessary algebraic manipulations become tedious, even though the technique is essentially the same as given in the main body of this paper. Thus, in this section only the results will be stated.

The determinant preceding equation 10 becomes

$$\begin{vmatrix} 1 - \dfrac{U}{c} & -\dfrac{A}{c} \\ (1 - \alpha)\dfrac{U}{A} - \dfrac{p(A)}{c} & 1 - \alpha\dfrac{U}{c} \end{vmatrix} = 0$$

which yields:

$$c = \alpha U \pm (\alpha(\alpha - 1) U^2 + Ap(A))^{1/2}$$

Recall that $UA = \kappa$ and it is also assumed $p = p(A)$, i.e., the relation between A and p is assumed to be more general than that previously used (equation 3).

The relation between slopes of the leading edges of the pressure pulse and velocity pulse (previously equation 11) is now

$$\frac{p_t}{U_t} = \frac{Ap'(A)}{c - U} \quad \text{(A11)}$$

The argument concerning leading edges is now slightly more complicated. Here it depends not only on stiffening of the tube but also on A. However, the conclusion remains the same; P_t is increasing and U_t is decreasing.

Equation 15 now becomes under this more general assumption

$$f_1(z)\frac{DU_t}{Dz} + f_2(z)U_t + f_3(z)U_t^2 = 0 \quad \text{(A15)}$$

where

$$f_1(z) = 2(c^2 - c\alpha U + 2\alpha U^2)$$

$$f_2(z) = c\left[\frac{p'(A)}{A}(c - 2U) + \alpha(c - U)\frac{DU}{Dz} + Ap''(A) + \frac{(c - 2U)(c - U)}{A}\frac{DA}{Dz} + \frac{c}{A}(c - U)(c - 2\alpha U + U)\frac{D}{Dz}\left(\frac{A}{c - U}\right)\right]$$

$$f_3 = \frac{(1 - \alpha)c}{c - U} - (\alpha c - 3\alpha U + 2c) + \frac{A^2p''(A)}{(c - U)}.$$

An equation analogous to equation 16 may be obtained by using equations A11 and A15.

It should be noted in passing that equation A15 may be used to provide physiological data on κ, γ, α, and $p(A)$. If it were possible to measure the slope of the leading edge of the velocity or pressure pulse at several points, then Bellman's technique of quasilinearization (10) could be applied to provide estimates of κ, γ, etc.

The numerical calculations reported in this paper were made at the Common Research Computer Facility of the Texas Medical Center. The Facility is supported by grants from the National Institutes of Health and other sources.

Received for publication 25 May 1966.

REFERENCES

1. McDonald, D. A., Blood Flow in Arteries, London, Edward Arnold (Publishers) Ltd., 1960, 258.
2. Womersley, J. R., An elastic tube theory of pulse transmission and oscillatory flow in mammalian arteries, Wright Air Development Center Technical Report TR 56-614, 1957, 54.
3. Peterson, L. H., *Circulation Research,* 1954, **2,** 127.
4. Starr, I., *J. Applied Physiol.,* 1957, **11,** 174.
5. Fox, E. A., and Saibel, E., *Trans. Soc. Rheology,* 1963, **7,** 25.
6. Varley, E., and Cumberbatch, E., *J. Inst. Maths. Applics.,* 1964, **1,** 101.
7. Barnard, A. C. L., Hunt, W. A., Timlake, W. E., and Varley, E., *Biophysic. J.,* 1966, **6,** 717.
8. Patel, D. J., Greenfield, J. C., Jr., and Fry, D. L., *in* Pulsatile Blood Flow, (E. O. Attinger, editor), New York, McGraw-Hill Book Company, 1964, 293.
9. Akers, W. W., Barnard, A. C. L., Bourland, H. M., Hunt, W. A., Timlake, W. P., and Varley, E., *Biophysic. J.,* 1966, **6,** 725.
10. Bellman, R., and Kalaban, R., Quasilinearization and Nonlinear Boundary Value Problems, New York, American Elsevier Publishing Co., Inc., 1965.